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Generalizations of the strong Ekeland variational principle with a generalized distance in complete metric spaces

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Abstract

In this paper, we prove a generalization of the strong Ekeland variational principle for a generalized distance (i.e., u -distance) on complete metric spaces. The result present in this paper extends and improves the corresponding result of Georgiev (J. Math. Anal. Appl. 131:1-21, 1988) and Suzuki (J. Math. Anal. Appl. 320:788-794, 2006).

Keywords: u -distance; complete metric space; Ekeland's variational principle

1 Introduction

In 1974, Ekeland [1] proved the following, which is called *the Ekeland variational principle* (for short, EVP).

Theorem 1.1 [1] *Let (X, d) be a complete metric space with metric d and f be a function from X into $(-\infty, +\infty]$ which is proper lower semicontinuous bounded from below. Then for $u \in X$ and $\lambda > 0$, there exists $v \in X$ such that*

$$(P) \quad f(v) \leq f(u) - \lambda d(u, v);$$

$$(Q) \quad f(w) > f(v) - \lambda d(v, w) \text{ for every } w \neq v.$$

Later, Takahashi [2] showed that this principle is equivalent to the Caristi fixed point theorem and nonconvex minimization theorem. In 1988, Georgiev [3] proved the following generalization of Theorem 1.1, which is called *the strong Ekeland variational principle*.

Theorem 1.2 [3] *Let X be a complete metric space with metric d and $f : X \rightarrow (-\infty, +\infty]$ be proper lower semicontinuous bounded from below. Then, for all $u \in X$, $\lambda > 0$ and $\delta > 0$, there exists $v \in X$ satisfying the following:*

$$(P)' \quad f(v) < f(u) - \lambda d(u, v) + \delta;$$

$$(Q) \quad f(w) > f(v) - \lambda d(v, w) \text{ for every } w \in X \setminus \{v\};$$

$$(R) \quad \text{if a sequence } \{x_n\} \text{ in } X \text{ satisfies } \lim_{n \rightarrow \infty} (f(x_n) + \lambda d(v, x_n)) = f(v), \text{ then } \{x_n\} \text{ converges to } v.$$

On the other hand, Kada *et al.* [4] introduced the concept of w -distance defined on a metric space and extended the Ekeland variational principle, the Kirk-Caristi fixed point theorem and the minimization theorem for w -distance. Recently, Suzuki [5, 6] introduced

a more general concept than w -distance, which is called τ -distance, and established the strong Ekeland variational principle for τ -distance. Very recently, Ume [7] introduced a more generalized concept than τ -distance, which is called u -distance, and proved a new minimization and a new fixed point theorem by using u -distance on a complete metric space.

In this paper, we prove the strong Ekeland variational principle for u -distance on a complete metric space. The results of this paper extend and generalize some results in Georgiev [3], Suzuki [5], Ansari [9] and Park [10].

2 Preliminaries

Throughout the paper, we denote by \mathbb{N} the set of all positive integers, by \mathbb{R} the set of real numbers, $\mathbb{R}_+ = [0, \infty)$. Let us recall the following well-known definition of a u -distance.

Definition 2.1 ([8] and [7]) Let X be a complete metric space with metric d . Then a function $p : X \times X \rightarrow \mathbb{R}_+$ is called a u -distance on X if there exists a function $\theta : X \times X \times [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}_+$ such that

- (u1) $p(x, z) \leq p(x, y) + p(y, z)$ for all $x, y, z \in X$;
- (u2) $\theta(x, y, 0, 0) = 0$, $\theta(x, y, s, t) \geq \min\{s, t\}$ for all $x, y \in X$ and $s, t \in [0, \infty)$, and for any $x \in X$ and for every $\epsilon > 0$, there exists $\delta > 0$ such that $|s - s_0| < \delta$, $|t - t_0| < \delta$, $s, s_0, t, t_0 \in [0, \infty)$ and $y \in X$ imply

$$|\theta(x, y, s, t) - \theta(x, y, s_0, t_0)| < \epsilon;$$

- (u3) $\lim_{n \rightarrow \infty} x_n = x$ and $\lim_{n \rightarrow \infty} \sup\{\theta(w_n, z_n, p(w_n, x_m), p(z_n, x_m)) : m \geq n\} = 0$ imply $p(y, x) \leq \lim_{n \rightarrow \infty} \inf p(y, x_n)$ for all $y \in X$;
- (u4) $\lim_{n \rightarrow \infty} \sup\{p(x_n, w_m) : m \geq n\} = 0$, $\lim_{n \rightarrow \infty} \sup\{p(y_n, z_m) : m \geq n\} = 0$, $\lim_{n \rightarrow \infty} \theta(x_n, w_n, s_n, t_n) = 0$ and $\lim_{n \rightarrow \infty} \theta(y_n, z_n, s_n, t_n) = 0$ imply $\lim_{n \rightarrow \infty} \theta(w_n, z_n, s_n, t_n) = 0$ or $\lim_{n \rightarrow \infty} \sup\{p(w_m, x_n) : m \geq n\} = 0$, $\lim_{n \rightarrow \infty} \sup\{p(z_m, y_n) : m \geq n\} = 0$, $\lim_{n \rightarrow \infty} \theta(x_n, w_n, s_n, t_n) = 0$ and $\lim_{n \rightarrow \infty} \theta(y_n, z_n, s_n, t_n) = 0$ imply $\lim_{n \rightarrow \infty} \theta(w_n, z_n, s_n, t_n) = 0$;
- (u5) $\lim_{n \rightarrow \infty} \theta(w_n, z_n, p(w_n, x_n), p(z_n, x_n)) = 0$ and $\lim_{n \rightarrow \infty} \theta(w_n, z_n, p(w_n, y_n), p(z_n, y_n)) = 0$ imply $\lim_{n \rightarrow \infty} d(x_n, y_n) = 0$ or $\lim_{n \rightarrow \infty} \theta(a_n, b_n, p(x_n, a_n), p(x_n, b_n)) = 0$ and $\lim_{n \rightarrow \infty} \theta(a_n, b_n, p(y_n, a_n), p(y_n, b_n)) = 0$ imply $\lim_{n \rightarrow \infty} d(x_n, y_n) = 0$.

Proposition 2.2 [7] Let p be a u -distance on a metric space (X, d) and c be a positive real number. Then a function $q : X \times X \rightarrow \mathbb{R}_+$ defined by $q(x, y) = c \cdot p(x, y)$ for every $x, y \in X$ is also a u -distance on X .

Lemma 2.3 [7] Let (X, d) be a metric space and let p be a u -distance on X . If $\{x_n\}$ is a p -Cauchy sequence, then $\{x_n\}$ is a Cauchy sequence.

Lemma 2.4 [7] Let (X, d) be a metric space and p be a u -distance on X . Suppose that a sequence $\{x_n\}$ of X satisfies

$$\lim_{n \rightarrow \infty} \sup\{p(x_n, x_m) : m > n\} = 0$$

or

$$\lim_{n \rightarrow \infty} \sup \{p(x_m, x_n) : m > n\} = 0.$$

Then, $\{x_n\}$ is a p -Cauchy sequence and $\{x_n\}$ is a Cauchy sequence.

3 Main theorem

Lemma 3.1 Let X be a complete metric space and p be a u -distance on X . If a sequence $\{x_n\}$ of X satisfies $\lim_{n \rightarrow \infty} p(z, x_n) = 0$ for some $z \in X$, then $\{x_n\}$ is a p -Cauchy sequence. Moreover, if a sequence $\{y_n\}$ of X also satisfies $\lim_{n \rightarrow \infty} p(z, y_n) = 0$, then $\lim_{n \rightarrow \infty} p(x_n, y_n) = 0$. In particular, for $x, y, z \in X$, $p(z, x) = 0$ and $p(z, y) = 0$ imply $x = y$.

Proof Let θ be a function from $X \times X \times [0, \infty) \times [0, \infty)$ into \mathbb{R}_+ satisfying (u1)-(u5). From $\lim_n p(z, x_n) = 0$, it follows by (u2) that $\lim_{n \rightarrow \infty} \theta(z, z, p(z, x_n), p(z, x_n)) = 0$. Therefore, $\{x_n\}$ is a p -Cauchy sequence. \square

Theorem 3.2 Let X be a complete metric space and T be a mapping from X into itself. Suppose that there exists a u -distance p on X and $r \in [0, 1)$ such that $p(Tx, T^2x) \leq r \cdot p(x, Tx)$ for all $x \in X$. Assume that either of the following hold:

- (i) If $\lim_{n \rightarrow \infty} \sup \{p(x_n, x_m) : m > n\} = 0$, $\lim_{n \rightarrow \infty} p(x_n, Tx_n) = 0$ and $\lim_{n \rightarrow \infty} p(x_n, y) = 0$, then $Ty = y$;
- (ii) if $\{x_n\}$ and $\{Tx_n\}$ converge to y , then $Ty = y$;
- (iii) T is continuous.

Then, there exists $x_0 \in X$ such that $Tx_0 = x_0$ and $p(x_0, x_0) = 0$.

Proof It is the same as the proof of Theorem 1 in [5]. \square

Lemma 3.3 Let X be a complete metric space, p be a u -distance on X and ϕ be a function from $X \times X$ into $(-\infty, \infty]$ satisfying

- (1) $\phi(x, z) \leq \phi(x, y) + \phi(y, z)$ for all $x, y, z \in X$;
- (2) $\phi(x, \cdot) : X \rightarrow (-\infty, \infty]$ is lower semicontinuous for any $x \in X$;
- (3) there exists an x_0 such that $\inf_{y \in X} \phi(x_0, y) > -\infty$; and
- (4) $\phi(x, y) = -\phi(y, x)$.

Define $Mx = \{y \in X : \phi(x, y) + p(x, y) \leq 0\}$. Let $u \in X$ and $c \in \mathbb{R}_+$ such that $\phi(x, u) < \infty$ for all $x \in X$, $Mu \neq \emptyset$ and $c \geq \phi(x, u) - \inf_{y \in Mu} \phi(u, y)$. Then a function $q : X \times X \rightarrow \mathbb{R}_+$ defined by

$$q(x, y) = \begin{cases} \phi(u, x) - \inf_{y \in Mx} \phi(u, y) & \text{if } x \in Mu \text{ and } y \in Mx, \\ c + p(x, y) & \text{if } x \notin Mu \text{ or } y \notin Mx \end{cases}$$

is a u -distance on X .

Proof Let η be a function from $X \times X \times \mathbb{R}_+ \times \mathbb{R}_+$ into \mathbb{R}_+ satisfying (u2)-(u5) for a u -distance. We note that $\phi(x, y) + \phi(y, z) + p(x, y) + p(y, z) \leq 0$ and $\phi(x, z) + p(x, z) \leq 0$. Thus, $y \in Mx$ and $z \in My$ imply $z \in Mx$. If $x \in Mu$ and $y \in Mx$, then

$$\begin{aligned} p(x, y) &\leq \phi(y, x) \leq q(x, y) = \phi(y, x) - \inf_{y \in Mx} \phi(x, y) \\ &\leq \phi(x, u) - \inf_{y \in Mu} \phi(x, y) \leq c. \end{aligned}$$

Therefore, $p(x, y) \leq q(y, x) \leq c + p(x, y)$ for all $x, y \in X$. To complete the proof, we will show (u1)_q, (u3)_{q,η}, (u4)_{q,η} and (u5)_{q,η}. Let x, y and z be fixed elements in X . In the case $x \in Mu$, $y \in Mx$, $y \in Mu$ and $z \in My$, we have $z \in Mx$ and hence $q(x, z) = q(x, y) \leq q(x, y) + q(y, z)$. In the other case, we note that

$$\begin{aligned} q(x, z) &\leq c + p(x, z) \leq c + p(x, y) + p(y, z) \\ &\leq 2c + p(x, y) + p(y, z) \\ &= q(x, y) + q(y, z). \end{aligned}$$

This shows (u1)_q.

We next suppose that $\lim_{n \rightarrow \infty} x_n = x$ and $\lim_{n \rightarrow \infty} \sup\{\eta(w_n, z_n, q(w_n, x_m), q(z_n, x_m)) : m \geq n\} = 0$ and fix $w \in X$. Since $\lim_{n \rightarrow \infty} \sup\{\theta(w_n, z_n, p(w_n, x_m), p(z_n, x_m)) : m \geq n\} = 0$, we have $p(w, x) \leq \liminf_{n \rightarrow \infty} p(w, x_n)$ for all $y \in X$.

In the case that $w \in Mu$ and there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \in Mw$ for all $k \in \mathbb{N}$, we have

$$\begin{aligned} \phi(w, x) + p(w, x) &\leq \liminf_{n \rightarrow \infty} \phi(w, x_n) + \lim_{n \rightarrow \infty} p(w, x_n) \\ &\leq \liminf_{n \rightarrow \infty} (\phi(w, x_n) + p(w, x_n)) \\ &\leq \liminf_{k \rightarrow \infty} (\phi(w, x_{n_k}) + p(w, x_{n_k})) \\ &\leq 0, \end{aligned}$$

and so $x \in Mu$. Hence

$$q(w, x) = \phi(u, w) - \inf_{x \in Mw} \phi(u, x) = \lim_{k \rightarrow \infty} q(w, x_{n_k}) = \liminf_{n \rightarrow \infty} q(w, x_n).$$

In the other case, we obtain

$$\begin{aligned} q(w, x) &\leq c + p(w, x) \leq \liminf_{n \rightarrow \infty} (c + p(w, x_n)) \\ &= \liminf_{n \rightarrow \infty} q(w, x_n). \end{aligned}$$

This shows (u3)_{q,η}. We will show that q satisfies (u4)_{q,η}.

Case I: Suppose that $\lim_{n \rightarrow \infty} \sup\{q(x_n, w_m) : m \geq n\} = 0$, $\lim_{n \rightarrow \infty} \sup\{q(y_n, z_m) : m \geq n\} = 0$, $\lim_{n \rightarrow \infty} \eta(x_n, w_n, s_n, t_n) = 0$, and $\lim_{n \rightarrow \infty} \eta(y_n, z_n, s_n, t_n) = 0$.

In the case $x_n \in Mu$ and $w_m \in Mx_n$, we note that $q(x_n, w_m) = \phi(u, x_n) - \inf_{w_m \in Mx_n} \phi(u, w_m)$. Since $\phi(x_n, w_m) + p(x_n, w_m) \leq 0$, it follows that

$$\begin{aligned} p(x_n, w_m) &\leq -\phi(x_n, w_m) = \phi(w_m, x_n) \\ &\leq \phi(w_m, u) + \phi(u, x_n) \\ &= \phi(u, x_n) - \phi(u, w_m) \\ &\leq \phi(u, x_n) - \inf_{w_m \in Mx_n} \phi(u, w_m) = q(x_n, w_m). \end{aligned}$$

Thus, we have $p(x_n, w_m) \leq q(x_n, w_m)$. This implies that $\sup_{m \geq n} p(x_n, w_m) \leq \sup_{m \geq n} q(x_n, w_m)$. Take $n \rightarrow \infty$, so

$$0 \leq \lim_{n \rightarrow \infty} \sup p(x_n, w_m) \leq \lim_{n \rightarrow \infty} \sup q(x_n, w_m) = 0$$

and therefore $\lim_{n \rightarrow \infty} \sup p(x_n, w_m) = 0$.

Similarly, if $y_n \in Mu$ and $z_m \in My_n$, then $\lim_{n \rightarrow \infty} \sup p(y_n, z_m) = 0$.

We note that $\lim_{n \rightarrow \infty} \theta(x_n, w_n, s_n, t_n) = 0 = \lim_{n \rightarrow \infty} \theta(y_n, z_n, s_n, t_n)$ and hence

$$\lim_{n \rightarrow \infty} \eta(w_n, z_n, s_n, t_n) = 0.$$

In the case $x_n \neq Mu$ or $w_m \neq Mx_n$, we note that $p(x_n, w_m) \leq c + p(x_n, w_m) = q(x_n, w_m)$. Thus, we have $p(x_n, w_m) \leq q(x_n, w_m)$. This implies that $\sup_{m \geq n} p(x_n, w_m) \leq \sup_{m \geq n} q(x_n, w_m)$. Taking $n \rightarrow \infty$, we obtain

$$0 \leq \lim_{n \rightarrow \infty} \sup p(x_n, w_m) \leq \lim_{n \rightarrow \infty} \sup q(x_n, w_m) = 0$$

and therefore $\lim_{n \rightarrow \infty} \sup p(x_n, w_m) = 0$. Similarly as above, if $y_n \neq Mu$ and $z_m \neq My_n$, then $\lim_{n \rightarrow \infty} \sup p(y_n, z_m) = 0$. We note that $\lim_{n \rightarrow \infty} \theta(x_n, w_n, s_n, t_n) = 0 = \lim_{n \rightarrow \infty} \theta(y_n, z_n, s_n, t_n)$ and hence $\lim_{n \rightarrow \infty} \eta(w_n, z_n, s_n, t_n) = 0$.

Case II: Suppose that $\lim_{n \rightarrow \infty} \sup \{q(w_m, x_n) : m \geq n\} = 0$, $\lim_{n \rightarrow \infty} \sup \{q(z_m, y_n) : m \geq n\} = 0$, $\lim_{n \rightarrow \infty} \eta(x_n, w_n, s_n, t_n) = 0$ and $\lim_{n \rightarrow \infty} \eta(y_n, z_n, s_n, t_n) = 0$. Similarly as in Case I, we can show that $\lim_{n \rightarrow \infty} \eta(w_n, z_n, s_n, t_n) = 0$. This shows (u4)_{q,η}. We will show that q satisfies (u5)_{q,η}.

Case I: Suppose that $\lim_{n \rightarrow \infty} \eta(w_n, z_n, q(x_n, w_n), q(x_n, z_n)) = 0$ and $\lim_{n \rightarrow \infty} \eta(w_n, z_n, q(y_n, w_n), q(y_n, z_n)) = 0$. In the case $x_n \in Mu$ and $w_n, z_n \in Mx_n$, we note that $q(x_n, w_n) = \phi(u, x_n) - \inf_{w_n \in Mx_n} \phi(u, w_n)$ and hence $q(x_n, z_n) = \phi(u, x_n) - \inf_{z_n \in Mx_n} \phi(u, z_n)$. Thus, we have

$$\begin{aligned} \theta(w_n, z_n, p(x_n, w_n), p(x_n, z_n)) &\leq \theta(w_n, z_n, \phi(z_n, x_n), \phi(z_n, x_n)) \\ &\leq \theta(w_n, z_n, \phi(w_n, u) + \phi(u, x_n), \phi(z_n, u) + \phi(u, x_n)) \\ &= \theta(w_n, z_n, \phi(u, x_n) - \phi(u, w_n), \phi(u, x_n) - \phi(u, z_n)) \\ &\leq \theta\left(w_n, z_n, \phi(u, x_n) - \inf_{w_n \in Mx_n} \phi(u, w_n), \phi(u, x_n) - \inf_{z_n \in Mx_n} \phi(u, z_n)\right) \\ &= \eta(w_n, z_n, q(x_n, w_n), q(x_n, z_n)). \end{aligned}$$

Taking $n \rightarrow \infty$, we have

$$0 \leq \lim_{n \rightarrow \infty} \theta(w_n, z_n, p(x_n, w_n), p(x_n, z_n)) \leq \lim_{n \rightarrow \infty} \eta(w_n, z_n, q(x_n, w_n), q(x_n, z_n)) = 0.$$

Therefore $\lim_{n \rightarrow \infty} \theta(w_n, z_n, p(x_n, w_n), p(x_n, z_n)) = 0$. Similarly, if $y_n \in Mu$ and $z_n, w_n \in My_n$, then $\lim_{n \rightarrow \infty} \theta(w_n, z_n, p(y_n, w_n), p(y_n, z_n)) = 0$. In the case $x_n \neq Mu$ or $w_n, z_n \neq Mx_n$, we have $q(x_n, w_n) = c + p(x_n, w_n)$ and $q(x_n, z_n) = c + p(x_n, z_n)$. Since p is a u -distance, we have

$\lim_{n \rightarrow \infty} d(x_n, y_n) = 0$. Hence

$$\begin{aligned} \theta(w_n, z_n, c + p(x_n, w_n), c + p(x_n, z_n)) &\leq \theta(w_n, z_n, c + p(x_n, w_n), c + p(x_n, z_n)) \\ &\leq \eta(w_n, z_n, q(x_n, w_n), q(x_n, z_n)). \end{aligned}$$

Take $n \rightarrow \infty$, thus

$$0 \leq \lim_{n \rightarrow \infty} \theta(w_n, z_n, p(x_n, w_n), p(x_n, z_n)) \leq \lim_{n \rightarrow \infty} \eta(w_n, z_n, q(x_n, w_n), q(x_n, z_n)) = 0.$$

Therefore $\lim_{n \rightarrow \infty} \theta(w_n, z_n, p(x_n, w_n), p(x_n, z_n)) = 0$. Similarly, if $y_n \neq Mu$ or $w_n, z_n \neq My_n$, then $\lim_{n \rightarrow \infty} \theta(w_n, z_n, p(y_n, w_n), p(y_n, z_n)) = 0$. Since p is a u -distance, we have $\lim_{n \rightarrow \infty} d(x_n, y_n) = 0$.

Case II: Suppose that $\lim_{n \rightarrow \infty} \eta(w_n, z_n, q(w_n, x_n), q(z_n, x_n)) = 0$ and $\lim_{n \rightarrow \infty} \eta(w_n, z_n, q(w_n, y_n), q(z_n, y_n)) = 0$. Similarly as in Case I, we can show that $\lim_{n \rightarrow \infty} d(x_n, y_n) = 0$. This shows (u5)_{q,η}. \square

Proposition 3.4 Let X be a complete metric space, p be a u -distance on X and ϕ be a function from $X \times X$ into $(-\infty, \infty]$ satisfying

- (1) $\phi(x, z) \leq \phi(x, y) + \phi(y, z)$ for all $x, y, z \in X$;
- (2) $\phi(x, \cdot) : X \rightarrow (-\infty, \infty]$ is lower semicontinuous for any $x \in X$;
- (3) there exists an x_0 such that $\inf_{y \in X} \phi(x_0, y) > -\infty$; and
- (4) $\phi(x, y) = -\phi(y, x)$.

Define $Mx = \{y \in X : \phi(x, y) + p(x, y) \leq 0\}$ for all $x \in X$. Then, for each $u \in X$ with $Mu \neq \emptyset$, there exists $x_0 \in Mu$ such that $Mx_0 \subset \{x_0\}$. In particular, there exists $y_0 \in X$ such that $My_0 \subset \{y_0\}$.

Proof Let $u \in X$ with $Mu \neq \emptyset$. We have $u_1 \in Mu$ by $\phi(u, u_1) < \infty$. If $Mu = \emptyset$, the assertion holds. Suppose that $Mu_1 \neq \emptyset$ and $Mx \cap (X \setminus \{x\}) \neq \emptyset$ for all $x \in Mu_1$. Let $u_2 \in Mu_1$. We know that $\phi(x, y) \leq 0$ for all $x \in X$ and $y \in Mx$, we define a mapping $T : X \rightarrow X$ as follows: For each $x \in Mu_1$, Tx satisfies $Tx \in Mx$, $Tx \neq x$ and

$$\phi(u_1, Tx) \leq \frac{\phi(u_1, x) + \inf_{y \in Mx} \phi(u_1, y)}{2}.$$

For each $x \notin Mu_1$, define $Tx = u_2 \neq x$. We also define a function $q : X \times X \rightarrow \mathbb{R}^+$ by

$$q(x, y) = \begin{cases} \phi(u, x) - \inf_{y \in Mx} \phi(u_1, y) & \text{if } x \in Mu_1 \text{ and } y \in Mx, \\ 2\phi(u, u_1) - 2\inf_{w \in Mu_1} \phi(u, w) + 1 + p(x, y) & \text{if } x \notin Mu_1 \text{ or } y \notin Mx. \end{cases}$$

By Lemma 3.3, we have q is a u -distance on X . Since $y \in My$ and $z \in My$, it follows by Lemma 3.3 that $z \in Mx$. Hence $Tx \in Mu_1$ and $MTx \subset Mx$ for all $x \in Mu_1$. If $x \in Mu_1$, we obtain

$$\begin{aligned} q(Tx, T^2x) &= \phi(u_1, Tx) - \inf_{y \in MTx} \phi(u_1, y) \\ &\leq \frac{\phi(u_1, x) + \inf_{y \in Mx} \phi(u_1, y)}{2} - \inf_{y \in Mx} \phi(u_1, y) \\ &= \frac{q(x, Tx)}{2}. \end{aligned}$$

If $x \notin Mu_1$,

$$\begin{aligned} q(Tx, T^2x) &= q(u_2, Tu_2) = \phi(u_1, u_2) - \inf_{Tu_2 \in Mu_2} \phi(u_1, Tu_2) \\ &\leq \phi(u, u_1) - \inf_{Tu_1} \phi(u, Tu_1) \\ &\leq \frac{q(x, u_2)}{2} = \frac{q(x, Tx)}{2}. \end{aligned}$$

We will show (i) in Theorem 3.2. Suppose that $\lim_{n \rightarrow \infty} \sup\{q(x_n, x_m) : m > n\} = 0$ and $\lim_{n \rightarrow \infty} q(x_n, y) = 0$. We may assume $x_n \in Mu_1$ and $y \in Mx_n$ for all $n \in \mathbb{N}$ by the definition of q . Then $y \in Mu_1$ and hence $Ty \in My \subset Mx_n$. By Lemma 2.4 we have $\lim_{n \rightarrow \infty} q(x_n, Ty) = \lim_{n \rightarrow \infty} q(x_n, y) = 0$ and $Ty = y$. Hence, by Theorem 3.2, T has a fixed point. This is a contradiction. So, there is $x_0 \in Mu_1 \subset Mu$ such that $Mx_0 \subset \{x_0\}$. \square

Theorem 3.5 *Let X be a complete metric space, p be a u -distance on X and ϕ be a function from $X \times X$ into $(-\infty, \infty]$ satisfying*

- (1) $\phi(x, z) \leq \phi(x, y) + \phi(y, z)$ for all $x, y, z \in X$;
- (2) $\phi(x, \cdot) : X \rightarrow (-\infty, \infty]$ is lower semicontinuous for any $x \in X$;
- (3) there exists an x_0 such that $\inf_{y \in X} \phi(x_0, y) > -\infty$; and
- (4) $\phi(x, y) = -\phi(y, x)$.

Then the following hold:

- (A) *For each $u \in X$, there exists $v \in X$ such that $\phi(u, v) \leq 0$ and $\phi(v, w) + p(v, w) > 0$ for all $w \in X \setminus \{v\}$;*
- (B) *For each $\lambda > 0$ and $u \in X$ with $p(u, u) = 0$, there exists $v \in X$ such that $\phi(u, v) + \lambda p(u, v) \leq 0$ and $\phi(v, w) + \lambda p(v, w) > 0$ for all $w \in X \setminus \{v\}$.*

Proof We will show that (A). For each $x \in X$, we define Mx as in Proposition 3.4. If $Mu = \emptyset$, we have u that satisfies $\phi(u, w) + p(u, w) > 0$ for all $w \in X$ with $w \neq u$. If $Mu \neq \emptyset$ and there exists $v \in Mu$, then it follows by Proposition 3.4 that $Mv \subset \{v\}$. Since $v \in Mu$ implies $\phi(u, v) \leq 0$ and $Mv \subset \{v\}$, this shows that $\phi(v, w) + p(v, w) > 0$ for all $w \in X$ with $w \neq v$.

We will show that (B). By Proposition 2.2, we note that λp is a u -distance. We define $Mx = \{y \in X : \phi(x, y) + \lambda p(x, y) \leq 0\}$ for all $x \in X$. Since $p(u, u) = 0$, we have $Mu \neq \emptyset$, and hence there exists $v \in Mu$ such that $Mv \subseteq \{v\}$ by Proposition 3.4. Therefore v satisfies $\phi(u, v) + \lambda p(u, v) \leq 0$ and $\phi(v, w) + \lambda p(v, w) > 0$ for all $w \in X$ with $w \neq v$. This completes the proof. \square

Remark 3.6 By setting $\phi(x, y) = f(y) - f(x)$, where $f : X \rightarrow \mathbb{R}$ is lower semicontinuous bounded below, and letting p be a τ -distance in Theorem 3.5, we obtain the Ekeland variational principle proved by Suzuki [5].

Theorem 3.7 *Let X be a complete metric space, p be a u -distance on X and ϕ be a function from $X \times X$ into $(-\infty, \infty]$ satisfying*

- (1) $\phi(x, z) \leq \phi(x, y) + \phi(y, z)$ for all $x, y, z \in X$;
- (2) $\phi(x, \cdot) : X \rightarrow (-\infty, \infty]$ is lower semicontinuous for any $x \in X$;
- (3) there exists an x_0 such that $\inf_{y \in X} \phi(x_0, y) > -\infty$; and
- (4) $\phi(x, y) = -\phi(y, x)$.

Let $u \in X$ with $p(u, u) = 0$. Then $\lambda > 0$ and $\delta > 0$, there exists $v \in X$ satisfying the following:

- (i) $\phi(u, v) \leq 0$;
- (ii) $\phi(u, v) + \lambda p(u, v) < \delta$;
- (iii) $\phi(v, w) + \lambda p(v, w) > 0$ for all $w \in X \setminus \{v\}$;
- (iv) if a sequence $\{x_n\}$ in X satisfies $\lim_n(\phi(v, x_n) + \lambda p(v, x_n)) = 0$, then $\{x_n\}$ is p -Cauchy, $\lim_n x_n = v$ and $p(v, v) = \lim_n p(v, x_n) = 0$.

Proof In the case $\phi(v, u) = \infty$, (i) and (ii) hold for all $v \in X$. We also note that (iii) and (iv) do not depend on $\phi(v, u)$. In the case $\phi(v, u) < \infty$, set $\lambda' \in (0, \lambda)$ satisfying

$$\frac{\lambda - \lambda'}{\lambda'} \left(\phi(u, v) - \inf_{x \in X} \phi(v, x) \right) < \delta.$$

By Theorem 3.5(B), there exists $v \in X$ such that $\phi(u, v) + \lambda' p(u, v) \leq 0$ and $\phi(v, w) + \lambda' p(v, w) > 0$ for all $w \in X \setminus \{v\}$. Thus, we have

$$\begin{aligned} \phi(u, v) &= \left(1 + \frac{\lambda - \lambda'}{\lambda'} \right) \phi(u, v) - \left(\frac{\lambda - \lambda'}{\lambda'} \right) \phi(u, v) \\ &\leq \left(1 + \frac{\lambda - \lambda'}{\lambda'} \right) (-\lambda' p(u, v)) - \left(\frac{\lambda - \lambda'}{\lambda'} \right) \phi(u, v) \\ &= -\lambda' p(u, v) - (\lambda - \lambda') (p(u, v)) - \left(\frac{\lambda - \lambda'}{\lambda'} \right) \phi(u, v) \\ &= -\lambda p(u, v) - \left(\frac{\lambda - \lambda'}{\lambda'} \right) \phi(u, v) \\ &\leq -\lambda p(u, v) + \left(\frac{\lambda - \lambda'}{\lambda'} \right) \phi(u, v) + \left(\frac{\lambda - \lambda'}{\lambda'} \right) \inf_{x \in X} \phi(x, v) \\ &= -\lambda p(u, v) + \left(\frac{\lambda - \lambda'}{\lambda'} \right) \left(\phi(u, v) - \inf_{x \in X} \phi(v, x) \right) \\ &< -\lambda p(u, v) + \delta. \end{aligned}$$

Therefore, $\phi(u, v) + \lambda p(u, v) < \delta$. For $w \in X \setminus \{v\}$, we note that

$$\phi(v, w) > -\lambda' p(v, w) \geq -\lambda p(v, w).$$

So, $\phi(v, w) + \lambda p(v, w) > 0$. Finally, we will show that (iv). Suppose that a sequence $\{x_n\}$ in X satisfies $\lim_n(\phi(v, x_n) + \lambda p(v, x_n)) = 0$. We note that $\phi(v, w) + \lambda' p(v, w) \geq 0$ for all $w \in X$. We have

$$\begin{aligned} \lim_{n \rightarrow \infty} \sup p(v, x_n) &= \lim_{n \rightarrow \infty} \sup \left(\frac{\lambda - \lambda'}{\lambda - \lambda'} \right) p(v, x_n) \\ &= \lim_{n \rightarrow \infty} \frac{\lambda p(v, x_n) - \lambda' p(v, x_n)}{\lambda - \lambda'} \\ &\leq \lim_{n \rightarrow \infty} \frac{\lambda p(v, x_n) - \phi(v, x_n)}{\lambda - \lambda'} \\ &\leq \lim_{n \rightarrow \infty} \frac{\lambda p(v, x_n) + \phi(v, x_n)}{\lambda - \lambda'} \\ &= 0. \end{aligned}$$

By Lemma 3.1, $\{x_n\}$ is a p -Cauchy sequence. From Lemma 2.3, therefore $\{x_n\}$ is a Cauchy sequence. By the completeness of X , $\{x_n\}$ converges to some point $x \in X$. From (u3), we have $p(v, x) = 0$ and so

$$\begin{aligned}\phi(v, x) &\leq \liminf_{n \rightarrow \infty} \phi(v, x_n) \\ &\leq \lim_{n \rightarrow \infty} (\phi(v, x_n) + \lambda p(v, x_n)) = 0.\end{aligned}$$

Thus, if $v \neq x$, then we have

$$\phi(v, x) > -\lambda' p(v, x) \geq \phi(v, x).$$

This is a contradiction. Hence, we obtain $v = x$. □

Remark 3.8 By setting $\phi(x, y) = f(y) - f(x)$, where $f : X \rightarrow \mathbb{R}$ is lower semicontinuous bounded below. Let p be a τ -distance in Theorem 3.7, we obtain the strong Ekeland variational principle proved by Suzuki [6].

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors read and approved the final manuscript.

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